Optimum Two-Impulse Transfers for Preliminary Interplanetary Trajectory Design

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A method for numerically determining the optimum two-impulse transfer between two positions in two different heliocentric orbits is described. The formulas are based on earlier work by McCue and Bender for optimum two-impulse transfers between two orbits about the Earth. The contours of minimum total ΔV(ΔV to launch plus ΔV to achieve a given final orbit at the target body) are plotted on axes of true anomaly of the launch body at launch and true anomaly of the target body at arrival. The optimum transfers obtained are "time-open" and the ΔV from the initial orbit to the final orbit is found using patched conics. These "prime rib" plots are useful for selecting initial conditions in the early phases of mission design.

Nomenclature

f	= true anomaly
i	= inclination
I	= impulse vector
J	= total ΔV
p	= semilatus rectum
q	= radius of periapse
Q	= radius of apoapse
r	= magnitude of position vector
$\boldsymbol{r}$	= position vector
U	= unit vector in the direction of $\boldsymbol{r}$
V	= magnitude of velocity vector
$\boldsymbol{V}$	= velocity vector
ΔV	= change in velocity
μ	= gravitational parameter
θ	= transfer angle

Subscripts

t	= transfer trajectory
1	= launch body
2	= target body
$\infty 1$	= outgoing hyperbolic excess
$\infty 2$	= incoming hyperbolic excess

I. Introduction

THE problem of finding the optimum two-impulse transfers between any two inclined noncoapsidal elliptical orbits has been studied by McCue and Bender,[1,2] Jezewski,[3] and many others.[4] The solutions presented by McCue and Bender were given as contours of the minimum total ΔV to transfer between fixed positions on initial and final orbits about the Earth. The positions were specified by the angles from one of the common nodes. Similar contours of minimum total ΔV for optimum two-impulse transfers between bodies in elliptical heliocentric orbits have been suggested by Hulkower and Ross[5,6] as a tool in analyzing trajectories in the solar system. These curves are called "prime ribs" because of their crude resemblance to the popular entree. In Sec. II we present the analytic formulation of the results of Ref. 2 modified to include the gravitational fields of the launch and target bodies. Examples of prime rib plots and the tabular output generated by a computer program using these formulas are presented in Sec. III. Uses for prime ribs are suggested in Sec. IV.

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II. Analytical Formulation

Consider the problem of finding the minimum two-impulse transfer from a given parking orbit about one body to a final state in an orbit about the target body, as illustrated in Fig. 1. We assume that the time required to leave the parking orbit is negligible and the resulting heliocentric velocity is the vector sum of the orbital velocity of the launch body V_1 and the hyperbolic excess velocity $V_{\infty 1}$. If the parking orbit has a perigee q_1 and an apogee Q_1 then the change in velocity required at perigee to leave with the desired $V_{\infty 1}$ is

$$\Delta V_1 = [V_{\infty 1}^2 + (2\mu_1/q_1)]^{1/2} - \{2\mu_1 Q_1/[q_1(Q_1+q_1)]\}^{1/2}$$

The direction of the velocity $V_{\infty 1}$ is of course determined by the orientation of the parking orbit ellipse. It is assumed that the parking orbit may be oriented freely to obtain the desired direction.

Similarly, at the target body, we assume that the spacecraft is placed in an elliptical orbit by an impulse at periapse. Thus,

$$\Delta V_2 = [V_{\infty 2}^2 + (2\mu_2/q_2)]^{1/2} - \{2\mu_2 Q_2/[q_2(Q_2+q_2)]\}^{1/2}$$

We assume that the shapes, given by q_1 and Q_1, and q_2 and Q_2, of both departure and arrival orbits around the two bodies are fixed. Only the magnitudes of $V_{\infty 1}$ and $V_{\infty 2}$ are considered varying as the orbital positions change and as the time on the transfer varies. To conform with the notation of Ref. 2, we define

$$I_1 = V_{\infty 1} \quad \text{and} \quad I_2 = V_{\infty 2}$$

In this case, the quantity to be minimized is

$$J = \Delta V_1 + \Delta V_2 = [I_1 \cdot I_1 + (2\mu_1/q_1)]^{1/2} - \{2\mu_1 Q_1/[q_1(Q_1+q_1)]\}^{1/2} + [I_2 \cdot I_2 + (2\mu_2/q_2)]^{1/2} - \{2\mu_2 Q_2/[q_2(Q_2+q_2)]\}^{1/2}$$

If we fix the true anomaly of the launch body at launch f_1 and that of the arrival body f_2 then the one remaining variable in the problem may be taken, as in Ref. 2, to be the semilatus

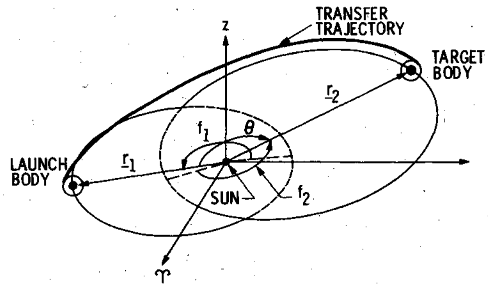

Fig. 1 Geometry of the problem.

rectum of the elliptical transfer orbit, p_t. Note that p_t is directly related to the time of flight. At the minimum of J we have

$$0=\frac{\partial J}{\partial p_t}=\frac{\partial \Delta V_1}{\partial p_t}+\frac{\partial \Delta V_2}{\partial p_t}=\frac{I_1\cdot(\partial I_1/\partial p_t)}{\Delta V_1+\{2\mu_1 Q_1/[q_1(Q_1+q_1)]\}^{1/2}}$$

$$+\frac{I_2\cdot(\partial I_2/\partial p_t)}{\Delta V_2+\{2\mu_2 Q/[q_2(Q_2+q_2)]\}^{1/2}}$$

Since the nomenclature describing the orbits of the initial body, target body, and transfer are the same as that of McCue and Bender, the following formulas, which describe the solution, are taken directly from Ref. 2. The departure and arrival velocities on the transfer orbit are

$$V_{t1}=\pm(v+zU_1)=I_1+V_1$$

$$V_{t2}=\pm(v-zU_2)=V_2-I_2$$

where the upper sign refers to a short (0 deg$<\theta<$180 deg) trajectory and the lower to a long (180 deg$<\theta<$360 deg) trajectory, and where

$$v=[(\mu p_t)^{1/2}(r_2-r_1)]/|r_1\times r_2| \quad \text{and} \quad z=(\mu/p_t)^{1/2}\tan(\theta/2)$$

The derivatives of I_1 and I_2 with respect to p_t are

$$\partial I_1/\partial p_t=\pm(1/2p_t)(v-zU_1) \quad \text{and} \quad \partial I_2/\partial p_t=\mp(1/2p_t)(v+zU_2)$$

The algorithm employed to find the value of p_t that satisfies the minimum condition for both long and short elliptical transfers starts by determining the range of possible values of p_t. The two parabolic limits of p_t are given by

$$p_{\min}=\frac{r_1r_2-r_1\cdot r_2}{r_1+r_2+[2(r_1r_2+r_1\cdot r_2)]^{1/2}}$$

$$p_{\max}=\frac{r_1r_2-r_1\cdot r_2}{r_1+r_2-[2(r_1r_2+r_1\cdot r_2)]^{1/2}}$$

The technique combines Newton's method with halving the search interval and requires only the analytic form of the first derivative.

In running the computer program for each pair of positions on the two orbits the optimum values of p_t are used to compute the two values of J, namely, J_{short} and J_{long}. Only the smaller of these is saved. The positions were specified by the true anomalies but can be specified just as well by mean anomalies on the two orbits. In addition, the time of flight is computed for each optimum transfer. Finally, for transfers near 180 deg, geometric constraints generally force large changes in inclination and, thus high impulses. Such cases are of no interest and are avoided by the program.

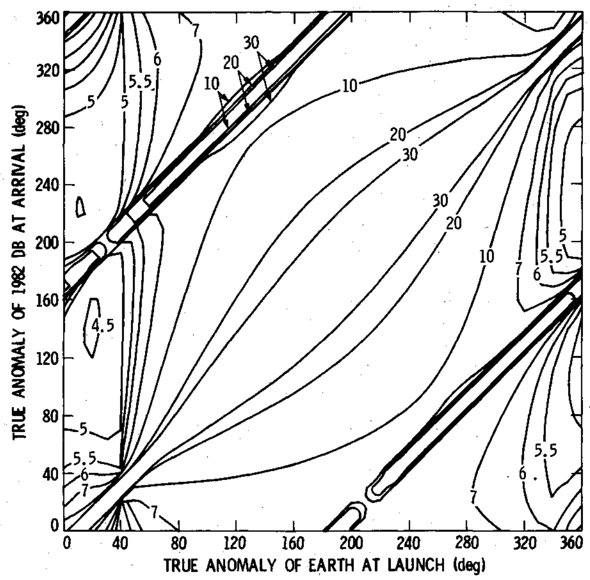

Fig. 2 Prime rib plot for the near-Earth asteroid 1982 DB. ΔV in km/s.

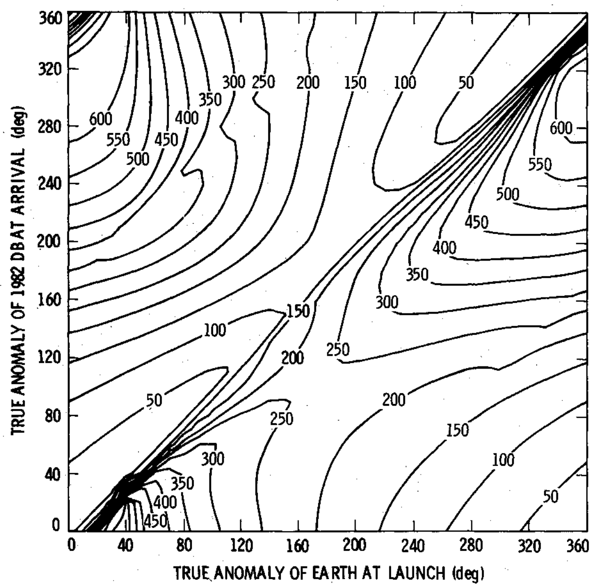

Fig. 3 Contours of time of flight in days corresponding to the optimal transfers in Fig. 2.

III. Prime Rib Plots and Tabular Output

Figures 2 and 3 display a prime rib plot and the corresponding time of flight contours for the optimum two-impulse transfers to rendezvous with the Earth-crossing asteroid, 1982 DB. The initial orbit about the Earth is the standard circular Shuttle orbit with a radius of 6656 km. The orbit at 1982 DB has a radius of periapse of 15 km and a radius of apoapse of 5×10^7 km. The minimum ΔV is computed every 10 deg in both f_1 at launch and f_2 at arrival.

By inspection one can see that the best transfers occur when the Earth is located at $f_1\sim$20 deg at launch and 1982 DB is located at 120 deg $\leq f_2\leq$160 deg when the spacecraft arrives. Table 1 covers the region containing the globally optimal transfer. The data displayed are the true and mean anomalies of Earth at launch; the true and mean anomalies of 1982 DB at arrival and its mean anomaly at launch; the minimum two-impulse total ΔV required of transfer from Earth to 1982 DB at each f_1 at launch and f_2 at arrival; the corresponding time of flight of each optimal transfer and the trajectory type. The

Table 1 Sample output from prime rib program

True anomaly of Earth at launch = 20.0000 deg
Mean anomaly of Earth at launch = 19.3526 deg

True anomaly of 1982 DB at arrival	Mean anomaly of 1982 DB at arrival	Mean anomaly of 1982 DB at launch	Minimum ΔV, km/s	Time of flight, days	Trajectory type
0.0000	0.0000	−272.8370	11.0625	503.1281	Long
10.0000	4.4002	−61.5759	58.6301	121.6640	Long
20.0000	8.8720	4.1982	10.3631	8.6187	Short
30.0000	13.4898	4.2214	7.0014	17.0916	Short
40.0000	18.3345	4.2356	5.9870	25.9991	Short
50.0000	23.4961	4.2366	5.4789	35.5156	Short
60.0000	29.0777	4.2194	5.1656	45.8403	Short
70.0000	35.1989	4.1825	4.9523	57.1961	Short
80.0000	41.9988	4.1267	4.7995	69.8384	Short
90.0000	49.6385	4.0531	4.6874	84.0622	Short
100.0000	58.3003	3.9624	4.6044	100.2023	Short
110.0000	68.1825	3.8545	4.5433	118.6248	Short
120.0000	79.4858	3.7282	4.4995	139.7017	Short
130.0000	92.3878	3.5816	4.4704	163.7640	Short
140.0000	107.0031	3.4119	4.4550	191.0284	Short
150.0000	123.3325	3.2164	4.4550	221.5014	Short
160.0000	141.2116	2.9956	4.4779	254.8788	Short
170.0000	160.2808	2.7669	4.5512	290.4653	Short
180.0000	180.0000	2.6634	4.8237	327.0196	Short
190.0000	199.7191	6.9158	33.4571	355.5411	Short
200.0000	218.7883	2.2781	4.7610	399.2582	Long
210.0000	236.6675	1.3834	4.5443	433.8782	Long
220.0000	252.9969	0.6226	4.5072	465.3936	Long
230.0000	267.6122	−0.1877	4.5136	493.8393	Long
240.0000	280.5141	−1.0903	4.5405	519.2958	Long
250.0000	291.8175	−2.0957	4.5819	541.9939	Long
260.0000	301.6997	−3.1910	4.6369	562.2371	Long
270.0000	310.3615	−4.3307	4.7071	580.3116	Long
280.0000	318.0012	−5.4152	4.7954	596.3996	Long
290.0000	324.8011	−6.2538	4.9069	610.4855	Long
300.0000	330.9223	−6.5042	5.0494	622.2352	Long
310.0000	336.5039	−5.5828	5.2356	630.8289	Long
320.0000	341.6655	−2.5406	5.4878	634.7371	Long
330.0000	346.5102	4.1086	5.8511	631.4094	Long
340.0000	351.1280	16.7258	6.4334	616.6582	Long
350.0000	355.5998	39.7777	7.5746	582.3952	Long
360.0000	360.0000	87.1630	11.0625	503.1281	Long

discontinuities from $f_2 = 10$ to 20 deg and at $f_2 = 190$ deg correspond to where there are switches from long to short and short to long trajectories for the optimal transfers. This switching occurs across the 180 deg transfer lines (see Fig. 2).

The program provides several options. Any object in the solar system in an elliptical heliocentric orbit can be either the launch or target body. Satellites of planets are precluded. Also, the range of values of the two true anomalies considered can be set by the user. This permits blowing up regions of interest. The net size is determined by the user as well. A flyby option may be invoked to optimize only the launch ΔV required to inject into a transfer orbit flying by a target body. In this mode the minimum single-impulse flyby problem is solved.

IV. Some Uses for Prime Ribs

In this section we discuss various ways prime ribs can be used in the early stages of mission design. We emphasize that prime ribs display the optimal two-impulse transfer between any f_1 at launch and any f_2 at arrival and are "time-open" or ephemeris-free solutions. In reality, only a one-dimensional subset of this plane occurs in any calendar year.[6] The globally optimal transfer may not be realized for many years for a given pair of bodies.

Many trajectories currently being designed include one or more additional impulses of a few hundred meters per second and/or a powered or unpowered swingby of a large third body. Although the prime ribs present solutions to the simpler two-impulse problem, they afford the mission designer a convenient means of selecting initial conditions for more sophisticated optimizers that can include additional impulses and gravitational assistance. To identify real opportunities to near-Earth asteroids we selected the initial positions and time of flight for the trajectory optimizer near the global optimum on the prime rib. In most cases, excellent candidates for missions resulted. The size of the region containing the global minimum is a very important factor in finding real trajectories in any given year. In some cases, the optimizer would add an additional impulse to lower the total ΔV. We concluded that prime ribs, by giving a picture of the entire space of minimum ΔV transfers, eliminated the guesswork in selecting initial conditions for the trajectory optimizer searching for real opportunities.

Once a real trajectory is identified it can be plotted on the prime rib.[6] By examining the width of the contours containing this trajectory, one can roughly determine the maximum size of the launch period before doing a more extensive and expensive search.

The flexibility afforded by the program in picking the launch and target bodies allows the generation of prime ribs for optimal return trajectories. These may be used in conjunction with the prime rib for direct trajectories to design sample return missions. The required approximate stay time can be determined by examining the height of the contour containing the Earth-to-target trajectory on the prime rib for this leg.

An interesting application of prime ribs is the determination of a figure to rank near-Earth asteroids by accessibility. This figure is the global minimum total ΔV for a two-impulse transfer of less than 360 deg from standard Shuttle orbit about the Earth to rendezvous with the object. The prime rib is used to identify the region in which to search for this global minimum. Initial conditions are fed into a trajectory optimizer with an ephemeris-free mode that finds the exact value by varying not only the time of flight or semilatus rectum of the transfer, but also f_1 at launch and f_2 at arrival. For 1982 DB, this figure is 4.453 km/s, located at (19°.0, 145°.1). Since this is the lowest figure found for any known near-Earth asteroid, 1982 DB is considered the most accessible of these objects. Near-optimal trajectories to rendezvous with 1982 DB launch in January 1981 (total ΔV=4.686 km/s) and in January 2002 (total ΔV=4.783 km/s).[7]

V. Conclusion

The example presented in this paper involved missions to a near-Earth asteroid. Prime ribs are certaintly very helpful in identifying trajectories to these objects. Since these plots can be generated for any pair of bodies in heliocentric orbit, their usefulness is limited only by the imagination of the mission designer. When one looks at a prime rib plot, he views the entire space of the best two-impulse transfer trajectories between two bodies. The information gleaned from this vantage has been valuable in finding real candidates for missions.

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